THE BINOMIAL THEOREM

1. Recall $0! = 1$, by definition, $3! = 3 \cdot 2 \cdot 1 = 6$ and $2! = 2 \cdot 1 = 2$. Hence, $\frac{3!\,2!}{0!} = \frac{(6)(2)}{1} = 12$.

2. We have $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ so $\frac{7!}{5!} = \frac{5040}{120} = 42$.

   While this is correct, we note that we could have saved ourselves some of time had we approached the problem as follows:

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42$$

   In fact, should we want to fully exploit the recursive nature of the factorial, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

3. Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

4. This problem continues the theme which we have seen in the previous two problems. We first note that since $k+2$ is larger than $k-1$, $(k+2)!$ contains all of the factors of $(k-1)!$ and as a result we can get the $(k-1)!$ to cancel from the denominator.

   To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k-1)$: $(k+2)! = (k+2)(k+1)(k)(k-1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

   The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

5. There are $\binom{70}{5} = 12,103,014$ different ways to select 5 distinct numbers from 1-70.

6. For each choice of numbers 1-70, there are 25 different tickets we could make, so there are a total of $25\binom{70}{5} = 302,575,350$ different tickets.

   NOTE: This matches the odds of winning as found on: http://www.megamillions.com/how-to-play

   beginenumerate

7. Since $(x-2)^4 = (x+(-2))^4$, we identify $a = x$, $b = -2$ and $n = 4$ and obtain

$$\begin{aligned}
(x-2)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j}(-2)^j \\
&= \binom{4}{0} x^{4-0}(-2)^0 + \binom{4}{1} x^{4-1}(-2)^1 + \binom{4}{2} x^{4-2}(-2)^2 + \binom{4}{3} x^{4-3}(-2)^3 + \binom{4}{4} x^{4-4}(-2)^4 \\
&= x^4 - 8x^3 + 24x^2 - 32x + 16
\end{aligned}$$

8. Writing $2.1^3 = (2+0.1)^3$. Identifying $a=2$, $b=0.1$ and $n=3$, we get

$$\begin{aligned}
(2+0.1)^3 &= \sum_{j=0}^{3} \binom{3}{j} 2^{3-j}(0.1)^j \\
&= \binom{3}{0}2^{3-0}(0.1)^0 + \binom{3}{1}2^{3-1}(0.1)^1 + \binom{3}{2}2^{3-2}(0.1)^2 + \binom{3}{3}2^{3-3}(0.1)^3 \\
&= 8 + 1.2 + 0.06 + 0.001 \\
&= 9.261
\end{aligned}$$

9. Identifying $a = 2x$, $b = y$ and $n = 5$, the Binomial Theorem gives

$$(2x+y)^5 = \sum_{j=0}^{5} \binom{5}{j} (2x)^{5-j} y^j$$

Since we are concerned with only the term containing x^3, we just need to locate the term which produces the x^3. The exponent on x is $5-j$ so if $5-j=3$ then $j=2$. Hence, the term we want is:

$$\binom{5}{2}(2x)^{5-2}y^2 = 10(2x)^3y^2 = 80x^3y^2$$

10. To get 'at least two' fours, we add the probabilities of obtaining exactly 2 fours, 3 fours, 4 fours, and 5 fours. Working through the details, we get:

$$\begin{aligned}
\text{Probability of at least two fours} &= \underbrace{\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{5-2}}_{\text{probability of 2 fours}} + \underbrace{\binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{5-3}}_{\text{probability of 3 fours}} \\
&\quad + \underbrace{\binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{5-4}}_{\text{probability of 4 fours}} + \underbrace{\binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^{5-5}}_{\text{probability of 5 fours}} \\
&= \tfrac{736}{3888} \approx 20\%
\end{aligned}$$

11. The sixth row of Pascal's Triangle is: 1 5 10 10 5 1.

12.

$$\begin{aligned}
(x+2y)^5 &= (1)(x)^5(2y)^0 + 5(x)^4(2y)^1 + 10(x)^3(2y)^2 + 10(x)^2(2y)^3 + 5(x)^1(2y)^4 + (2y)^5 \\
&= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5
\end{aligned}$$